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APPROXIMATE SOLUTION OF AN AXISYMMETRIC CONTACT PROBLEM WITH ALLOWANCE FOR TANGENTIAL DISPLACEMENTS ON THE CONTACT SURFACE

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A structurally nonlinear contact problem of a punch shaped like a paraboloid of revolution is studied. An equation for the contact-pressure density is derived with allowance for the radial tangential displacements of the boundary points of an elastic half-space. A method for constructing a closed-form approximate solution is proposed. The effect of the tangential displacements on the main contact parameters is discussed.

Key words: *contact problem, punch, contact pressure, approximate solution.*

1. Refined Formulation of the Contact Problem. We consider an elastic half-space $z \geqslant 0$ indented by a punch shaped like a body of revolution. Introducing a cylindrical coordinate system (r, φ, z), we write the equations of the punch surface (before loading):

$$z = -\Phi(r).$$

For simplicity, we assume that the punch occupies a convex domain $z \leqslant -\Phi(r)$ and it is in contact with the plane $z = 0$ at a single point chosen as the coordinate origin.

We denote the vertical displacement of the punch by δ_0. The condition that the points of the elastic body do not penetrate into the punch is written as follows [1] (see also [2]):

$$u_z(r, \varphi, 0) - \delta_0 + \Phi(r + u_r(r, \varphi, 0)) \geqslant 0. \tag{1.1}$$

The equality in relation (1.1) defines the contact area ω. Because of the axial symmetry and adopted shape of the punch, the area ω is a circle, whose radius is denoted by a. Thus, within the contact area ω, the following equation holds:

$$u_z - \delta_0 + \Phi(r + u_r) = 0 \qquad (r \leqslant a). \tag{1.2}$$

Here the displacements u_z and u_r depend only on the radius r.

We assume that the displacement u_r is small compared to the radius of the contact area ω. In this case, the nonlinear equation (1.2) can be replaced by the following linearized equation [1]:

$$u_z - \delta_0 + \Phi(r) + \Phi'(r) u_r = 0 \qquad (r \leqslant a). \tag{1.3}$$

Here and below, the prime denotes differentiation.

In the particular case of a punch shaped like a paraboloid of revolution

$$\Phi(r) = r^2/(2R_0)$$

(R_0 is the curvature radius of the punch surface at its apex), Eq. (1.3) becomes

$$u_z(r) + r u_r(r)/R_0 = \delta_0 - r^2/(2R_0) \qquad (r \leqslant a). \tag{1.4}$$

Expressing the displacements $u_z(r)$ and $u_r(r)$ in terms of the contact pressure $p(r)$ according to the solution of the Boussinesq problem (see, e.g., [3]), we write the displacement-compatibility condition (1.4) in the following form [4]:

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$$\iint\limits_{\omega} \frac{p(\rho)\,d\sigma}{R(r;\rho,\varphi)} - \frac{\alpha r}{2R_0} \iint\limits_{\omega} \frac{r-\rho\cos\varphi}{R(r;\rho,\varphi)^2}\, p(\rho)\,d\sigma = \frac{\pi E}{1-\nu^2}\Big(\delta_0 - \frac{r^2}{2R_0}\Big). \tag{1.5}$$

Here $d\sigma = \rho\,d\rho\,d\varphi$ is the unit area, E is Young's modulus, ν is Poisson's ratio, $R(r;\rho,\varphi) = (r^2+\rho^2-2r\rho\cos\varphi)^{1/2}$, and $\alpha = (1-2\nu)/(1-\nu)$.

Equation (1.5) is used to find the contact-pressure density $p(r)$. The radius of the contact area a is determined from the condition that the contact pressure is positive and vanishes at the edge of the contact area:

$$p(r) \geqslant 0 \quad (r \leqslant a), \qquad p(a) = 0. \tag{1.6}$$

A numerical solution of the problem considered was obtained in [1, 4]. A two-dimensional contact problem in a refined formulation was studied analytically in [5, 6]. Linearized conditions similar to (1.4) were considered for plate and shell models in [7, Sec. 2.3]. In the present paper, a method of constructing an approximate closed-form solution of Eq. (1.5) is proposed.

2. Equation for the Radius of the Contact Area. Using formula (3.613.2) of [8], we find

$$\int\limits_0^{2\pi}\int\limits_0^{a} \frac{r-\rho\cos\varphi}{r^2+\rho^2-2r\rho\cos\varphi}\, p(\rho)\rho\,d\rho\,d\varphi = \frac{2\pi}{r}\int\limits_0^{r} p(\rho)\rho\,d\rho. \tag{2.1}$$

With allowance for this equality, Eq. (1.5) becomes

$$\iint\limits_{\omega} \frac{p(\rho)\,d\sigma}{R(r;\rho,\varphi)} = u(r); \tag{2.2}$$

$$u(r) = \frac{\pi E}{1-\nu^2}\Big(\delta_0 - \frac{r^2}{2R_0}\Big) + \frac{\pi\alpha}{R_0}\int\limits_0^{r} p(\rho)\rho\,d\rho. \tag{2.3}$$

We use the general solution of the integral equation (2.2) obtained in [9–11]:

$$p(r) = \frac{F(a)}{\pi\sqrt{a^2-r^2}} - \frac{1}{\pi}\int\limits_r^{a} \frac{F'(s)}{\sqrt{s^2-r^2}}\,ds; \tag{2.4}$$

$$\pi F(r) = u(0) + r\int\limits_0^{r} \frac{u'(t)}{\sqrt{r^2-t^2}}\,dt. \tag{2.5}$$

Condition (1.6) for vanishing of the contact pressure at the edge of the contact area implies the equality $F(a) = 0$. According to (2.3) and (2.5), we have

$$\delta_0 = \frac{a^2}{R_0} - \frac{(1-2\nu)(1+\nu)}{ER_0}\, a\int\limits_0^{a} \frac{p(t)t}{\sqrt{a^2-t^2}}\,dt. \tag{2.6}$$

In this case, formula (2.4) becomes

$$p(r) = -\frac{1}{\pi}\int\limits_r^{a} \frac{F'(s)}{\sqrt{s^2-r^2}}\,ds. \tag{2.7}$$

Equation (2.6) is used to determine the unknown radius of the contact area from a specified value of the punch displacement δ_0.

3. Calculation of the Force that Presses the Punch against an Elastic Body. We denote the resultant of the contact pressure by P:

$$P = 2\pi\int\limits_0^{a} p(\rho)\rho\,d\rho. \tag{3.1}$$

Substitution of (2.7) into (3.1) leads to the following equation [11]:

$$P = 2\int_0^a F(s)\,ds. \tag{3.2}$$

Substituting the expression for the function $F(s)$ (2.5) into (3.2) and using (2.3), we have

$$u(0) = \frac{\pi E}{1-\nu^2}\,\delta_0, \qquad u'(t) = \frac{\pi\alpha}{R_0}\,p(t)t - \frac{\pi E}{1-\nu^2}\,\frac{t}{R_0}.$$

Thus, after evaluation of the definite integrals, Eq. (3.2) becomes

$$P = \frac{2E}{1-\nu^2}\Big(a\delta_0 - \frac{a^3}{3R_0}\Big) + \frac{2\alpha}{R_0}\int_0^a\int_0^s \frac{p(t)ts}{\sqrt{s^2-t^2}}\,dt\,ds.$$

Changing the order of integration in the iterated integral, we find that

$$P = \frac{2E}{1-\nu^2}\Big(a\delta_0 - \frac{a^3}{3R_0} + \frac{(1-2\nu)(1+\nu)}{ER_0}\int_0^a p(t)\sqrt{a^2-t^2}\,t\,dt\Big).$$

Taking into account the expression for the quantity δ_0 (2.6), we finally obtain

$$P = \frac{4E}{3(1-\nu^2)}\,\frac{a^3}{R_0} - \frac{2\alpha}{R_0}\int_0^a \frac{p(t)t^3}{\sqrt{a^2-t^2}}\,dt. \tag{3.3}$$

It is worth noting that Eq. (3.3) can be derived directly from Eq. (2.2) taking into account (2.3) and using the Mossakovskii theorem [12].

4. Maximum Contact Pressure. Formula (2.7) implies the following expression for the maximum value of the contact pressure (at the center of the contact area):

$$p(0) = -\frac{1}{\pi}\int_0^a \frac{F'(s)}{s}\,ds. \tag{4.1}$$

Differentiation of expression (2.5) yields

$$\pi F'(r) = \int_0^r \frac{u'(t)+tu''(t)}{\sqrt{r^2-t^2}}\,dt.$$

Substituting this expression into (4.1), we arrive at the formula

$$-\pi^2 p(0) = \int_0^a \frac{ds}{s}\int_0^s \frac{u'(t)+tu''(t)}{\sqrt{s^2-t^2}}\,dt.$$

Changing the order of integration, we obtain

$$-\pi^2 p(0) = \int_0^a \Big(\frac{\pi}{2} - \arcsin\frac{t}{a}\Big)[u'(t)+tu''(t)]\,\frac{dt}{t}.$$

Finally, integration by parts yields

$$-\pi^2 p(0) = \int_0^a \Big(\frac{\pi}{2} - \arcsin\frac{t}{a}\Big)\frac{u'(t)}{t}\,dt + \int_0^a \frac{u'(t)\,dt}{\sqrt{a^2-t^2}}. \tag{4.2}$$

Formula (4.2) was derived under the assumption that $u'(0) = 0$.

Differentiation of expression (2.5) with allowance for (2.3) leads to

$$\frac{F'(s)}{s} = -\frac{2E}{(1-\nu^2)R_0} + \frac{\alpha}{R_0 s}\Big(2\int_0^s \frac{p(t)t}{\sqrt{s^2-t^2}}\,dt + \int_0^s \frac{p'(t)t^2}{\sqrt{s^2-t^2}}\,dt\Big).$$

Changing the order of the integration by parts, we obtain

$$\int_0^a \int_0^s \frac{p(t)t}{s\sqrt{s^2-t^2}}\,dt\,ds = \int_0^a p(t)\arccos\frac{t}{a}\,dt.$$

Similarly, integrating by parts, we find

$$\int_0^a \int_0^s \frac{p'(t)t^2}{s\sqrt{s^2-t^2}}\,dt\,ds = t\Big(\frac{\pi}{2} - \arcsin\frac{t}{a}\Big)p(t)\Big|_0^a - \int_0^a p(t)\Big(\arccos\frac{t}{a} - \frac{t}{\sqrt{a^2-t^2}}\Big)\,dt.$$

We note that the sufficient condition for vanishing of the double substitution is the boundedness of the density $p(t)$ within the contact area.

These relations and formula (4.1) can be combined to give

$$p(0) = \frac{2E}{\pi(1-\nu^2)}\frac{a}{R_0} - \frac{\alpha}{\pi R_0}\Big(\int_0^a p(t)\arccos\frac{t}{a}\,dt + \int_0^a \frac{p(t)t}{\sqrt{a^2-t^2}}\,dt\Big). \tag{4.3}$$

We note that the last integral in (4.3) can be eliminated by virtue of Eq. (2.6).

5. Approximate Solution of the Contact Problem in a Refined Formulation. It should be noted that the equations obtained above, in particular, Eqs. (2.6), (3.3), and (4.3) are exact equalities derived from the original equation (1.5) without any simplifications. The right sides of these equations comprise the corresponding expression obtained by Hertz theory (see, e.g., [13, 14]) and a correction that takes into account the effect of the tangential displacements.

According to Hertz theory, the contact pressure under a punch shaped like paraboloid of revolution is given by

$$p(r) = p_0\sqrt{1 - r^2/a^2} \tag{5.1}$$

(p_0 is the maximum contact pressure).

Inserting expression (5.1) into the right side of Eq. (4.3), we obtain an approximate equation for the quantity p_0. Evaluating the quadratures, we have

$$p_0 = \frac{2E}{\pi(1-\nu^2)}\frac{a}{R_0} - \frac{\alpha}{2\pi}\frac{a}{R_0}\Big(\frac{\pi^2}{4} + 2\Big)p_0.$$

This relation implies that

$$p_0 = \frac{2E}{\pi(1-\nu^2)}\frac{a}{R_0}\Big(1 + \frac{\alpha}{2\pi}\frac{a}{R_0}\Big(\frac{\pi^2}{4} + 2\Big)\Big)^{-1}. \tag{5.2}$$

At the same time, substitution of expression (5.1) into Eq. (5.2) yields

$$\delta_0 = \frac{a^2}{R_0} - \frac{(1-2\nu)(1+\nu)}{2ER_0}p_0a^2. \tag{5.3}$$

Equation (5.3) with equality (5.2) are used to approximately determine the radius a of the contact area for a specified value of the punch displacement δ_0. Thus, the normalized radius of the contact area $x = a/R_0$ is determined from the equation

$$\frac{\delta_0}{R_0} = x^2\Big(1 - \frac{\alpha x}{\pi + \alpha x(\pi^2/8 + 1)}\Big). \tag{5.4}$$

The values of the roots of Eq. (5.4) agree well with the numerical results of [4] obtained by approximating the integral operators in Eq. (1.5) by finite sums. For example, the relative error in determining the radius of the contact area a does not exceed 2% for $\nu = 0.375$ and $\lambda = \alpha(\delta_0/2R_0)^{1/2} = 0.5$.

In the case where the force P is specified, the equation for determining the radius of the contact area should be derived from Eq. (3.3) combined with (5.1) and (5.2). As a result, the force P is given by

$$P = \frac{4E}{3(1-\nu^2)}\frac{a^3}{R_0} - \frac{\alpha a^3 p_0}{2R_0}, \tag{5.5}$$

where the quantity p_0 is defined by formula (5.2). It follows from Eq. (5.5) that

$$\frac{3(1-\nu^2)}{4ER_0^2}P = x^3\Big(1 - \frac{(3\alpha/4)x}{\pi + \alpha x(\pi^2/8+1)}\Big). \tag{5.6}$$

Finally, an approximate expression for the contact-pressure density is obtained by substituting the following expression into (2.7):

$$F(r) = \frac{E}{1-\nu^2}\Big(\delta_0 - \frac{r^2}{R_0}\Big) + \frac{\alpha p_0 r}{4R_0 a}\Big(2ra + (a^2 - r^2)\ln\frac{a+r}{a-r}\Big),$$

where the quantity p_0 is defined by formula (5.2).

6. Asymptotic Behavior of the Approximate Solution for a Small Contact Area. Assuming that the ratio $x = a/R_0$ is small, from Eq. (5.4) we obtain the following relation with accuracy to terms of order x^2 compared to unity:

$$a = \sqrt{\delta_0 R_0}\Big(1 + \frac{\alpha}{2\pi}\sqrt{\frac{\delta_0}{R_0}}\Big). \tag{6.1}$$

With allowance for (6.1), Eq. (5.6) yields

$$P = \frac{4E\sqrt{R_0}}{3(1-\nu^2)}\,\delta_0^{3/2}\Big(1 + \frac{3\alpha}{4\pi}\sqrt{\frac{\delta_0}{R_0}}\Big). \tag{6.2}$$

Substituting (6.1) into (5.2) and ignoring higher-order terms, we obtain

$$p_0 = \frac{2E}{\pi(1-\nu^2)}\sqrt{\frac{\delta_0}{R_0}}\Big(1 - \frac{\alpha}{2\pi}\Big(\frac{\pi^2}{4}+1\Big)\sqrt{\frac{\delta_0}{R_0}}\Big). \tag{6.3}$$

The accuracy of formulas (6.1)–(6.3) increases as the ratio δ_0/R_0 decreases.

7. Discussion. The exact relations (2.6), (3.3), and (4.3) allow one to study the effect of the tangential displacements on the main contact parameters. For example, solution of the contact problem in the refined formulation (for specified value of the punch displacement δ_0) leads to an increase in the contact-area radius since Eq. (2.6) implies the equality

$$a = \sqrt{\delta_0 R_0}\Big(1 - \frac{(1-2\nu)(1+\nu)}{E}\int_0^1 \frac{p(a\tau)\tau}{\sqrt{1-\tau^2}}\,d\tau\Big)^{-1/2}. \tag{7.1}$$

It should be noted that according to condition (1.6), the contact-pressure density within the area is positive.

Furthermore, in accordance with formula (3.3), we obtain

$$P = \frac{4E}{3(1-\nu^2)}\frac{a^3}{R_0}\Big(1 - \frac{3(1-2\nu)(1+\nu)}{2E}\int_0^1 \frac{p(a\tau)\tau^3}{\sqrt{1-\tau^2}}\,d\tau\Big). \tag{7.2}$$

Substitution of the radius a defined by formula (7.1) into Eq. (7.2) yields

$$P = \frac{4E\sqrt{R_0}}{3(1-\nu^2)}\frac{1-(3/2)I_3(p)}{\Big(1-I_1(p)\Big)^{3/2}}\,\delta_0^{3/2}, \qquad I_k(p) = \frac{(1-2\nu)(1+\nu)}{E}\int_0^1 \frac{p(a\tau)\tau^k}{\sqrt{1-\tau^2}}\,d\tau.$$

By virtue of the inequality $I_3(p) < I_1(p)$, it follows that accounting for the tangential displacements in the contact problem leads to an increase in the force P for a specified displacement of the punch δ_0. Using (4.3), we finally obtain

$$p(0) = \frac{2E}{\pi(1-\nu^2)}\frac{a}{R_0}\Big(1 - \frac{1}{2}I_1(p) - \frac{(1-2\nu)(1+\nu)}{2E}\int_0^1 p(a\tau)\arccos\tau\,d\tau\Big).$$

Substitution of expression (7.1) into this equality yields

$$p(0) = \frac{2E}{\pi(1-\nu^2)}\sqrt{\frac{\delta_0}{R_0}}\frac{1}{(1-I_1(p))^{1/2}}\Big(1 - \frac{1}{2}I_1(p) - \frac{(1-2\nu)(1+\nu)}{2E}\int_0^1 p(a\tau)\arccos\tau\,d\tau\Big).$$

Thus, allowance for the tangential displacements decreases the maximum value of the contact pressure. It is worth noting that a decrease in the maximum value of the contact pressure $p(0)$ accompanied by an increase in the resultant of the contact pressure P is achieved by redistribution of contact pressure over larger area. We also note that the conclusions formulated above agree with the asymptotic formulas (6.1)–(6.3).

The tangential displacements $u_r(a)$ were calculated in [4]. According to (2.1) and (3.2), we have

$$u_r(a) = -\frac{(1-2\nu)(1+\nu)}{2\pi E}\,\frac{P}{a}. \tag{7.3}$$

Substitution of expressions (7.1) and (3.3) into formula (7.3) yields

$$u_r(a) = -\frac{2\alpha\delta_0}{3\pi}\,\frac{1-(3/2)I_3(p)}{1-I_1(p)}.$$

It should be noted that in the derivation of Eqs. (1.5), deformations were ignored. In other words, the radius of the contact between the surfaces of the elastic bodies due to deformation is equal to $a + u_r(a)$.

Galanov [15] showed that accounting for tangential displacements leads to a decrease in incompatible displacements, i.e., penetration of the points of the elastic half-space into the punch. However, the contact problem formulated with allowance for the tangential displacements becomes much more complicated. The approximate solution constructed simplifies calculations and allows one to estimate the effect of this factor on the main contact parameters.

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